Logistic growth: clearly explained![*]

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1 Logistic Equation

The logistic equation (sometimes called the Verhulst model or logistic growth curve) is a model of population growth first published by Pierre Verhulst (1845, 1847).

$$N'[t] = \frac{rN[t](K - N[t])}{K} \tag{1}$$

where r is the Malthusian parameter (rate of maximum population growth) and K is the so-called carrying capacity (i.e., the maximum sustainable population). Then, divide both sides by K,

$$\frac{N'[t]}{K} = \frac{rN[t]\left(1 - \frac{N[t]}{K}\right)}{K} \tag{2}$$

[*]The title of this note has been inspired by the wonderful Youtuber Josh Starmer of StatQuest.

Letting $x = \frac{N[t]}{K}$, we obtain

$$x'(t) = r\,x(t)(1 - x(t)) \tag{3}$$

Demonstration. From equation 1, we rewrite the corresponding equation is the so called logistic differential equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{4}$$

Analytic Solution. The logistic equation can be solved by separation of variables:

$$\int \frac{dN}{N\left(1 - \frac{N}{K}\right)} = \int r\,dt \tag{5}$$

In order to evaluate the left hand side, we write:

$$\frac{1}{N\left(1 - \frac{N}{K}\right)} = \frac{K}{N(K-N)} = \frac{1}{N} + \frac{1}{K-N} \tag{6}$$

hence

$$\int \frac{dN}{N} + \int \frac{dN}{K-N} = \int r\,dt \tag{7}$$

$$ln|N| - ln|K-N| = rt + C \tag{8}$$

$$ln\left|\frac{K-N}{N}\right| = -rt - C \tag{9}$$

$$\left|\frac{K-N}{N}\right| = e^{-rt-C} \tag{10}$$

$$\frac{K-N}{N} = Ae^{-rt} \qquad\qquad (A = \pm e^{-C}) \tag{11}$$

From here we get :

$$N = \frac{K}{1+Ae^{-rt}} \qquad \text{where} \qquad A = \frac{K-N_0}{N_0} = \frac{K}{N_0} - 1 \tag{12}$$

Note that N_0 is the initial value.

Dividing both sides by K and defining $x \equiv N/K$. We finally obtain,

$$x(t) = \frac{1}{1+Be^{-rt}} \qquad \text{where} \qquad B = \frac{1}{x_0} - 1 \tag{13}$$

The function x(t) is sometimes known as the sigmoid function. The following commands has been used to plot the continuous version of the logistic model in the Wolfram Language as described in equation 3. The Malthusian parameter is equal to 1 for pedagogical purposes.

```
logit = Plot[(1*t*(1 - t)), t, 0, 1]
```

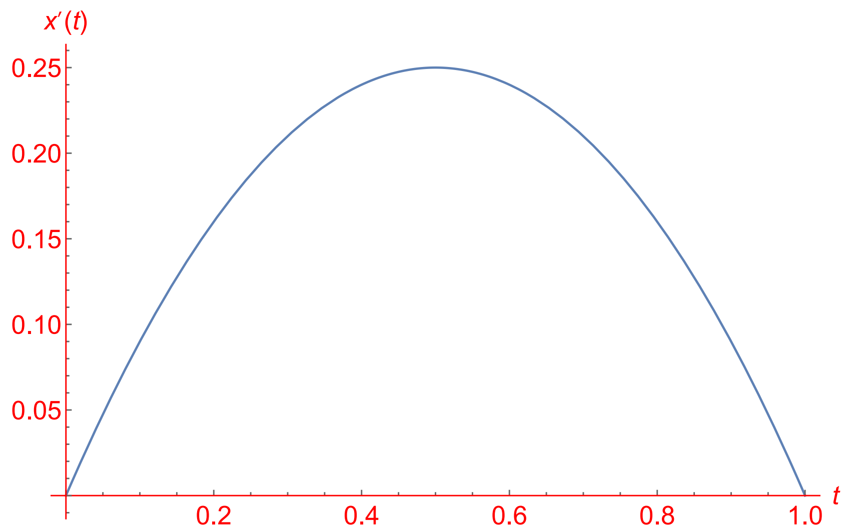

Figure 1: The logistic model

2 Sigmoid function

The logistic equation has a solution known has the sigmoid function.

The following commands has been used to plot the sigmoid function in the Wolfram Language as described in equation 13. The Malthusian parameter is equal to 1 for pedagogical purposes.

```
sig = Plot[ 1 / (1 + (1/0.05 - 1) / E^(1*t) ), 1 / ( 1 + ( 1 / 0.10 - 1) / E^(1*t) ) , t, 0, 10 ]
```

The plot of the solution is shown for initial conditions $x_0 = x(t = 0)$ ranging from 0.00 to 0.40 in steps of 0.05.

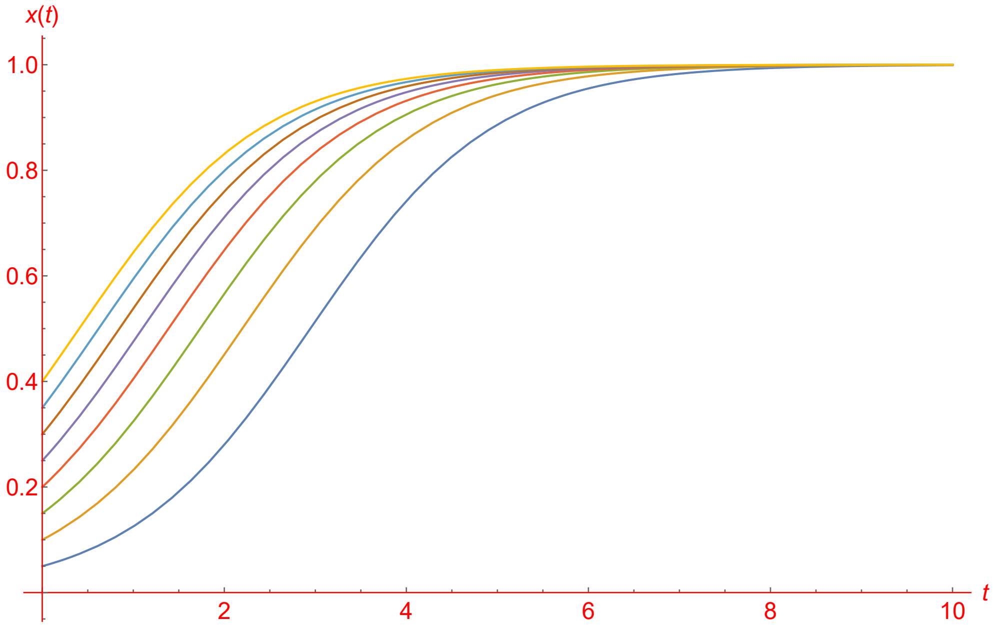

Figure 2: The sigmoid function

3 References

Weisstein, Eric W. 'Logistic Equation.' From MathWorld – A Wolfram Web Resource. Accessed 11 April 2020. Retrieved from URL.

Lerma, Miguel A. 'Notes on Calculus II.' Accessed 11 April 2020. Retrieved from URL.